J. Appl. & Pure Math. Vol. 8(2026), No. 3 - 4, pp. 243 - 260
https://doi.org/10.23091/japm.2026.243

HYPER G-MATRIX PAIRS, EXTENDED EIGENVALUE THEORY, AND EXTENDED SPECTRA IN INFINITE DIMENSIONS

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Abstract. This paper provides a detailed examination of the structural relationship between the concept of extended eigenvalues in Hilbert spaces and Hyper G-matrix pairs defined in matrix theory. We first present the definition of an extended eigenvalue, followed by the definition of a Hyper G-matrix pair. Subsequently, we establish the equivalence between the two structures, along with spectral connections and intertwining equalities at the operator level. In the final section, we systematically classify extended spectra in infinite-dimensional Hilbert spaces within the context of the Hyper G structure and thoroughly analyze the characteristic properties of extended spectra for special classes of operators such as normal, compact, hyponormal operators, and the Volterra operator.

AMS Mathematics Subject Classification : 47A10, 47A11, 47B20, 15A24, 15B99.
Key words and phrases : Hyper G-matrix pairs, extended eigenvalues, extended spectrum, intertwining operators, spectral theory, operator theory, Hilbert spaces.

1. Introduction

Let H be an infinite-dimensional, separable complex Hilbert space and let $L(H)$ denote the space of bounded linear operators. A complex number λ is called an *extended eigenvalue* of $A \in L(H)$ if there exists a nonzero operator T such that:

$$TA = \lambda AT. \tag{1}$$

In this case, T is called an *extended eigenoperator* of A. This concept is connected to Rosenblum's theorem, quantum mechanics, and the spectral analysis of integral operators [1, 12]. Biswas and Petrović [3] showed that

$$\sigma_{\text{ext}}(A) \subset \{\lambda \in \mathbb{C} : \sigma(A) \cap \sigma(\lambda A) \neq \emptyset\}.$$

Extended eigenvalues have been studied for various operator classes, including normal [9], quasinormal [13], and q-quasinormal operators [18]. The foundational work in functional analysis by Bachman and Narici [1] and perturbation theory

Received February 8, 2026. Revised April 6, 2026. Accepted April 14, 2026.

by Kato [12] provides the necessary background for these investigations. Brown [4] initiated studies on special operator classes, while Coburn [7] extended Weyl's theorem to nonnormal operators.

The study of operator equations of the form $AB = \lambda BA$ has been investigated by Cho et al. [6], revealing connections to extended eigenvalue problems. Karaev [11] examined extended eigenvalues for various operator classes, while Ismailov and Çevik [10] studied extended eigenvalues of direct sums of operators. Cassier and Alkanjo [5] explored extended spectra and extended eigenspaces of quasi-normal operators, providing important insights into the spectral structure of these operators.

The concept of Hyper G-matrix pairs has emerged as an important algebraic structure in matrix theory. Conway's work on subnormal operators [8] provides context for understanding operator-theoretic generalizations. Recent developments in Hyper G-matrix theory by Keleş [14, 15] and Keleş and Hall [19, 20] have revealed deep connections between matrix structures and operator theoretic concepts. In particular, Keleş and Hall [19] explored the fundamental properties distinguishing G-matrices from Hyper G-matrices, establishing important structural relationships. Their subsequent work [20] developed more sophisticated constructions of structured Hyper G-matrix pairs, providing a richer framework for understanding matrix factorizations and their operator theoretic implications.

The algebraic foundations of escort elements and intertwining structures have been developed by Keleş in his work on poloids and monoids [16, 17], which provides the categorical context for understanding the structural relationships between operators in extended eigenvalue problems.

The following definition establishes the fundamental concept of Hyper G-matrix pairs:

Definition 1.1. Let A and B be nonsingular square matrices. If there exist invertible diagonal matrices D_1 and D_2 such that

$$A^{-T} = D_1 B D_2 \tag{2}$$

then the pair (A, B) is called a *Hyper G-matrix pair*, denoted by $A \bowtie B$ [8, 14, 15].

Since D_1, D_2 are invertible and diagonal, equation (2) establishes a two-way scaled connection between A^{-T} and B.

The aim of this paper is to establish the natural connection between equations (1) and (2), reveal the meaning of this relationship in the context of operator theory, and systematically classify extended spectra in infinite dimensions within the framework of the Hyper G structure. Building upon the extensive body of work on extended eigenvalues [2, 3, 5, 6, 10, 11, 13, 18] and recent developments in Hyper G-matrix theory [14, 15, 19, 20], we develop a more comprehensive framework that connects matrix theory constructions with operator theoretic concepts. Our approach integrates insights from functional analysis [1, 12], operator theory [4, 7, 8, 9], and algebraic structures [16, 17] to provide a unified perspective on extended spectra.

TABLE 1. Core relationships between Hyper G-matrix pairs and extended eigenvalue theory

Concept	Hyper G-Matrix Pair	Extended Eigenvalue Theory
Basic Definition	$A^{-T} = D_1 B D_2$	$TA = \lambda AT$, $T \neq 0$
Notation	$A \bowtie B$	$\lambda \in \sigma_{\text{ext}}(A)$
Parameters	D_1, D_2 invertible diagonal	$\lambda \in \mathbb{C}$, $T \in L(H)$
Intertwining Structure	$D_2 A^T = B^{-1} D_1^{-1}$	$TA = A(\lambda T)$
Spectral Condition	$\sigma(A^{-T}) = \sigma(D_1 B D_2)$	$\sigma(A) \cap \sigma(\lambda A) \neq \emptyset$
Extended Eigenvalue	$\lambda_i = \dfrac{(B^{-1} D_1^{-1})_{ii}}{(D_2)_{ii}}$ (when A^T is diagonal)	λ values satisfying the equation
Special Case	$D_1 = \alpha I$, $D_2 = \beta I$ $A^{-T} = \alpha\beta B$	Trivial case: $\lambda = 1$ (usually)
Algebraic Structure	$\varepsilon_A = \{X : XA = AY \text{ for some } Y\}$	$\{T : TA = \lambda AT\}$ space
Dimension	Finite dimensional ($\mathbb{C}^{n\times n}$)	Infinite dimensional (Hilbert space)

Source: Prepared by the author.

Table 1 summarizes the fundamental similarities and differences between the two theories. As seen, both structures are shaped around intertwining relations and establish similar algebraic relationships between matrices and operators.

The following theorem establishes the fundamental algebraic consequences of the Hyper G-matrix pair definition:

Theorem 1.2. *Let $A \bowtie B$, i.e., there exist invertible diagonal matrices D_1, D_2 such that $A^{-T} = D_1 B D_2$. Then:*

(i) $A^T = D_2^{-1} B^{-1} D_1^{-1}$
(ii) $A^{-1} = D_2^T B^T D_1^T$
(iii) $B = D_1^{-1} A^{-T} D_2^{-1}$
(iv) $B^{-1} = D_2 A^T D_1$
(v) $B^{-T} = D_1 A D_2$

Proof. From $A^{-T} = D_1 B D_2$:

(i) Taking inverses: $A^T = (D_1 B D_2)^{-1} = D_2^{-1} B^{-1} D_1^{-1}$.
(ii) Taking transpose: $A^{-1} = (D_1 B D_2)^T = D_2^T B^T D_1^T$.
(iii) Solving for B: $B = D_1^{-1} A^{-T} D_2^{-1}$.
(iv) Taking inverse of (iii): $B^{-1} = D_2 A^T D_1$.
(v) Taking transpose of (iv): $B^{-T} = D_1 A D_2$.

□

We now formalize the definitions of extended eigenvalues and escort elements in operator theory:

Definition 1.3. Let $A \in L(H)$ be a bounded operator. If for $\lambda \in \mathbb{C}$ there exists a nonzero operator $T \in L(H)$ satisfying

$$TA = \lambda AT,$$

then λ is called an *extended eigenvalue* of A, and T is called the corresponding *extended eigenoperator*.

Definition 1.4. Let $A, B, C \in L(H)$. If $BA = AC$, then A is called an *escort element* between B and C [16, 17].

Equation (1) establishes an *intertwining* relation between the operators A and λA. This relation can be rewritten as:

$$TA = A(\lambda T).$$

Hence, in an equation of the form

$$XA = AY,$$

we can take $X = T$ and $Y = \lambda T$. Studying such operators algebraically, the set

$$\varepsilon_A = \{X \in L(H) : XA = AY \text{ for some } Y\}$$

forms an algebra.

Remark 1.1. The extended eigenvalue equation $TA = \lambda AT$ is essentially an intertwining relation between A and λA, and in this context T can be viewed as an escort element [16, 17].

To provide a simple and clear example of such an ε_A set, consider the following:

Example 1.5. On $H = \mathbb{C}^2$, consider the projection operator $A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$.

For $X = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in L(H)$, $XA = \begin{pmatrix} a & 0 \\ c & 0 \end{pmatrix}$ and $AY = \begin{pmatrix} p & q \\ 0 & 0 \end{pmatrix}$. For the equation $XA = AY$ to hold for some Y, it is necessary and sufficient that $c = 0$.

Thus,

$$\varepsilon_A = \left\{ \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} : a, b, d \in \mathbb{C} \right\},$$

which is a subalgebra of $L(H)$ and contains only those X satisfying $c = 0$, depending on the structure of A.

The following theorem establishes a direct link between Hyper G-matrix pairs and intertwining relations:

Theorem 1.6. *If $A \bowtie B$, then there exist diagonal matrices*

$$T = D_2 \quad \textit{and} \quad S = B^{-1}D_1^{-1}$$

such that

$$TA^T = S.$$

Proof. Under the assumption $A \bowtie B$, by definition there exist invertible diagonal matrices D_1 and D_2 such that $A^{-T} = D_1 B D_2$. Taking inverses gives $A^T = D_2^{-1}B^{-1}D_1^{-1}$ by Theorem 1.2. Multiplying both sides by D_2 yields:

$$D_2 A^T = B^{-1}D_1^{-1}.$$

Taking $T = D_2$ and $S = B^{-1}D_1^{-1}$ gives $TA^T = S$. □

To illustrate Theorem 1.6 with a concrete example, consider:

Example 1.7. Consider the following matrix:

$$A = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Since A is invertible,

$$A^{-T} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 1 & -1 & 1 & 0 \\ -1 & 1 & -1 & 1 \end{pmatrix}.$$

Now take the matrices

$$D_1 = \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 5 \end{pmatrix}, \qquad D_2 = \begin{pmatrix} 7 & 0 & 0 & 0 \\ 0 & 11 & 0 & 0 \\ 0 & 0 & 13 & 0 \\ 0 & 0 & 0 & 17 \end{pmatrix}.$$

Define $B := D_1^{-1} A^{-T} D_2^{-1}$. Hence $A \bowtie B$.
According to Theorem 1.6, taking $T = D_2$ and $S = B^{-1} D_1^{-1}$, we compute:

$$\begin{aligned} B^{-1} = D_2 A^T D_1 &= \begin{pmatrix} 7 & 0 & 0 & 0 \\ 0 & 11 & 0 & 0 \\ 0 & 0 & 13 & 0 \\ 0 & 0 & 0 & 17 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 5 \end{pmatrix} \\ &= \begin{pmatrix} 14 & 0 & 0 & 0 \\ 22 & 33 & 0 & 0 \\ 0 & 52 & 52 & 0 \\ 0 & 0 & 85 & 85 \end{pmatrix}. \end{aligned} \tag{3}$$

Then

$$\begin{aligned} S = B^{-1} D_1^{-1} &= \begin{pmatrix} 14 & 0 & 0 & 0 \\ 22 & 33 & 0 & 0 \\ 0 & 52 & 52 & 0 \\ 0 & 0 & 85 & 85 \end{pmatrix} \begin{pmatrix} 1/2 & 0 & 0 & 0 \\ 0 & 1/3 & 0 & 0 \\ 0 & 0 & 1/4 & 0 \\ 0 & 0 & 0 & 1/5 \end{pmatrix} \\ &= \begin{pmatrix} 7 & 0 & 0 & 0 \\ 11 & 11 & 0 & 0 \\ 0 & 13 & 13 & 0 \\ 0 & 0 & 17 & 17 \end{pmatrix}. \end{aligned} \tag{4}$$

We verify:

$$TA^T = D_2 A^T = \begin{pmatrix} 7 & 0 & 0 & 0 \\ 0 & 11 & 0 & 0 \\ 0 & 0 & 13 & 0 \\ 0 & 0 & 0 & 17 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix} = \begin{pmatrix} 7 & 0 & 0 & 0 \\ 11 & 11 & 0 & 0 \\ 0 & 13 & 13 & 0 \\ 0 & 0 & 17 & 17 \end{pmatrix} = S.$$

Thus Theorem 1.6 is verified on a concrete 4×4 example where D_1 and D_2 are diagonal, nonsingular, and distinct from the identity.

The next theorem shows how Hyper G pairs induce direct relationships between matrices:

Theorem 1.8. *If $A \bowtie B$, then there exist invertible diagonal matrices D_1, D_2 such that*

$$D_2 A^T = B^{-1} D_1^{-1}.$$

Moreover, if we denote $T = D_2$ and $S = B^{-1} D_1^{-1}$, then $TA^T = S$.

Proof. Since $A \bowtie B$, by definition there exist invertible diagonal matrices D_1 and D_2 such that

$$A^{-T} = D_1 B D_2.$$

Taking the inverse of both sides using Theorem 1.2:

$$A^T = D_2^{-1} B^{-1} D_1^{-1}.$$

Multiplying both sides by D_2:

$$D_2 A^T = B^{-1} D_1^{-1}.$$

□

From Theorem 1.8, we immediately obtain:

Corollary 1.9. *Under the Hyper G-matrix pair condition, the matrix A^T transforms under left multiplication by D_2 to give $B^{-1} D_1^{-1}$. This establishes a scaling relationship between A^T and the combination $B^{-1} D_1^{-1}$.*

This theorem reveals that the Hyper G-matrix pair definition naturally establishes direct relationships between matrices.

Let us demonstrate Theorem 1.8 with a numerical example:

Example 1.10. Let A and B be nonsingular 4×4 matrices, and D_1, D_2 nonsingular diagonal matrices. Consider the following matrices:

$$D_1 = \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 4 \end{pmatrix}, \qquad D_2 = \begin{pmatrix} 3 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, B = \begin{pmatrix} 1 & 2 & 0 & 1 \\ 0 & 1 & 3 & 0 \\ 1 & 0 & 1 & 2 \\ 2 & 1 & 0 & 1 \end{pmatrix}.$$

Define matrix A by

$$A = (D_1 B D_2)^{-T}$$

so that $A^{-T} = D_1 B D_2$, i.e., $A \bowtie B$.

Compute A^T:

$$A^T = D_2^{-1} B^{-1} D_1^{-1}.$$

Now compute both sides of $D_2 A^T = B^{-1} D_1^{-1}$:

Left side: $D_2 A^T = D_2(D_2^{-1} B^{-1} D_1^{-1}) = B^{-1} D_1^{-1}$.

Right side: $B^{-1} D_1^{-1}$.

Thus $D_2 A^T = B^{-1} D_1^{-1}$ holds identically by construction.

The connection between Hyper G pairs and extended eigenvalues is made explicit in the following proposition.

Proposition 1.11. *Let $A \bowtie B$ be a Hyper G-matrix pair. Then for the diagonal matrices $T = D_2$ and $S = B^{-1} D_1^{-1}$, the relation*

$$TA^T = S$$

holds. This implies that if A^T is diagonal (which is a consequence of the Hyper G structure), then the diagonal entries of A^T are given by $\lambda_i = s_i/t_i$, where t_i are the diagonal entries of T and s_i are the diagonal entries of S.

Proof. From Theorem 1.8, we have $TA^T = S$ with $T = D_2$ and $S = B^{-1}D_1^{-1}$. Let $T = \mathrm{diag}(t_1, \ldots, t_n)$ and $S = \mathrm{diag}(s_1, \ldots, s_n)$.

If A^T is diagonal, say $A^T = \mathrm{diag}(a_1, \ldots, a_n)$, then:

$$TA^T = \mathrm{diag}(t_1a_1, \ldots, t_na_n) = \mathrm{diag}(s_1, \ldots, s_n) = S.$$

Thus $t_ia_i = s_i$ for $i = 1, \ldots, n$, so $a_i = s_i/t_i$.

To see why A^T must be diagonal under the Hyper G condition, consider the equation $TA^T = S$. Since both T and S are diagonal, the (i, j)-entry of TA^T is $t_i(A^T)_{ij}$. The (i, j)-entry of S is $s_i\delta_{ij}$. Thus for $i \neq j$, we have $t_i(A^T)_{ij} = 0$, and since $t_i \neq 0$ (as T is invertible), we get $(A^T)_{ij} = 0$. Therefore A^T is diagonal. □

Remark 1.2. This proposition shows that given a Hyper G-matrix pair $A \bowtie B$, the matrix A^T must be diagonal. The diagonal entries give us the ratios:

$$(A^T)_{ii} = \frac{s_i}{t_i} = \frac{(B^{-1}D_1^{-1})_{ii}}{(D_2)_{ii}}.$$

These ratios can be interpreted in the context of extended eigenvalues when considering appropriate operator equations.

Thus, from a Hyper G-matrix pair, if we consider the equation $XA^T = A^TY$, we can take $X = T^{-1}$ and $Y = \mathrm{diag}(s_1/t_1, \ldots, s_n/t_n)$, giving:

$$T^{-1}A^T = A^T\mathrm{diag}(s_1/t_1, \ldots, s_n/t_n).$$

This provides an intertwining relation similar to the extended eigenvalue equation.

The spectral properties of Hyper G-matrix pairs are closely related to the spectral characterization of extended eigenvalues.

The following proposition presents the fundamental relationship between spectra in the case $A \bowtie B$:

Proposition 1.12. *Let $A \bowtie B$. Then:*

$$\sigma(A^{-T}) = \sigma(D_1BD_2).$$

However, in general $\sigma(D_1BD_2) \neq \sigma(B)$; since D_1 and D_2 are diagonal matrices, there is a scaling relation between $\sigma(D_1BD_2)$ and $\sigma(B)$.

Proof. By definition of $A \bowtie B$, we have $A^{-T} = D_1BD_2$. Since these two matrices are equal, their spectra coincide: $\sigma(A^{-T}) = \sigma(D_1BD_2)$.

Since D_1 and D_2 are general diagonal matrices, D_1BD_2 may not be similar to B. Therefore $\sigma(D_1BD_2) = \sigma(B)$ is not generally true. For example, take $B = I$ (identity matrix) and $D_1 = D_2 = \mathrm{diag}(2, 1)$; then $\sigma(D_1ID_2) = \{4, 1\}$ while $\sigma(I) = \{1, 1\}$. □

Let us illustrate Proposition 1.12 with a concrete example.

Example 1.13. Consider the following matrices:

$$B = \begin{pmatrix} 2 & 1 & 0 & 0 \\ 0 & 3 & 1 & 0 \\ 0 & 0 & 4 & 1 \\ 0 & 0 & 0 & 5 \end{pmatrix}, \quad D_1 = \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 5 \end{pmatrix}, \quad D_2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 4 \end{pmatrix}.$$

Since B is upper triangular, its eigenvalues are its diagonal entries: $\sigma(B) = \{2, 3, 4, 5\}$.

Compute $D_1 B D_2$:

$$\begin{aligned} D_1 B D_2 &= \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 5 \end{pmatrix} \begin{pmatrix} 2 & 1 & 0 & 0 \\ 0 & 3 & 1 & 0 \\ 0 & 0 & 4 & 1 \\ 0 & 0 & 0 & 5 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 4 \end{pmatrix} \\ &= \begin{pmatrix} 4 & 4 & 0 & 0 \\ 0 & 18 & 9 & 0 \\ 0 & 0 & 48 & 16 \\ 0 & 0 & 0 & 100 \end{pmatrix}. \end{aligned}$$

This matrix is also upper triangular, so its eigenvalues are its diagonal entries: $\sigma(D_1 B D_2) = \{4, 18, 48, 100\}$.

Thus we see:

$$\sigma(B) = \{2, 3, 4, 5\} \neq \{4, 18, 48, 100\} = \sigma(D_1 B D_2).$$

In this example, because B is upper triangular and D_1, D_2 are diagonal, $D_1 B D_2$ is also upper triangular and its diagonal entries are $(D_1)_{ii} \cdot b_{ii} \cdot (D_2)_{ii}$:

$$\begin{aligned} 4 &= 2 \cdot 2 \cdot 1, \\ 18 &= 3 \cdot 3 \cdot 2, \\ 48 &= 4 \cdot 4 \cdot 3, \\ 100 &= 5 \cdot 5 \cdot 4. \end{aligned}$$

This example shows that $\sigma(D_1 B D_2) = \sigma(B)$ is not generally true, but when B is upper triangular (or diagonal), the eigenvalues are scaled by the diagonal matrices.

The next proposition further clarifies the spectral relationships.

Proposition 1.14. *Let $A \bowtie B$. Then:*

$$\sigma(A^{-T}) = \sigma(D_1 B D_2).$$

However, in general $\sigma(D_1 B D_2) \neq \sigma(B)$. For $D_1 B D_2$ to be similar to B, D_1 and D_2 must have a special structure (e.g., both being scalar matrices).

Remark 1.3. If in a Hyper G-matrix pair the matrices D_1 and D_2 are scalar, i.e., $D_1 = \alpha I$ and $D_2 = \beta I$ ($\alpha, \beta \in \mathbb{C} \setminus \{0\}$), then the Hyper G condition becomes

$$A^{-T} = (\alpha I) B (\beta I) = \alpha\beta B.$$

In this case B is a scalar multiple of A^{-T}.

At the spectral level:

$$\sigma(A^{-T}) = \alpha\beta \cdot \sigma(B),$$

i.e., the spectra of A^{-T} and B are related by a scalar factor.

The matrix A^T becomes:

$$A^T = (\alpha\beta)^{-1} B^{-1}.$$

For the relation in Theorem 1.6, we have:

$$T = D_2 = \beta I, \quad S = B^{-1} D_1^{-1} = B^{-1}(\alpha^{-1} I) = (\alpha\beta)^{-1} A^T.$$

Then the relation $TA^T = S$ becomes:

$$\beta I \cdot A^T = (\alpha\beta)^{-1} A^T,$$

which implies $\beta A^T = (\alpha\beta)^{-1} A^T$, or equivalently, $\alpha\beta^2 = 1$.

This situation shows how rich and nontrivial the general Hyper G structure (D_1 and D_2 general diagonal matrices) can be, while the scalar case imposes additional constraints.

To illustrate the scalar case, consider the following example.

Example 1.15. Consider the following matrices:

$$A^{-T} = \begin{pmatrix} 2 & 1 & 0 & 0 \\ 0 & 3 & 1 & 0 \\ 0 & 0 & 4 & 1 \\ 0 & 0 & 0 & 5 \end{pmatrix}, \quad \alpha = 2, \quad \beta = 3.$$

Scalar matrices: $D_1 = 2I$, $D_2 = 3I$.

By the Hyper G condition:

$$B = \frac{1}{\alpha\beta} A^{-T} = \frac{1}{6} \begin{pmatrix} 2 & 1 & 0 & 0 \\ 0 & 3 & 1 & 0 \\ 0 & 0 & 4 & 1 \\ 0 & 0 & 0 & 5 \end{pmatrix} = \begin{pmatrix} \frac{1}{3} & \frac{1}{6} & 0 & 0 \\ 0 & \frac{1}{2} & \frac{1}{6} & 0 \\ 0 & 0 & \frac{2}{3} & \frac{1}{6} \\ 0 & 0 & 0 & \frac{5}{6} \end{pmatrix}.$$

Spectral check:

$$\sigma(A^{-T}) = \{2, 3, 4, 5\}, \quad \sigma(B) = \left\{ \frac{1}{3}, \frac{1}{2}, \frac{2}{3}, \frac{5}{6} \right\}.$$

Indeed $\sigma(A^{-T}) = 6 \cdot \sigma(B)$, i.e., scaling by factor $\alpha\beta = 6$.

The matrix A^T is the inverse of A^{-T}:

$$A^T = \begin{pmatrix} \frac{1}{2} & -\frac{1}{6} & \frac{1}{24} & -\frac{1}{120} \\ 0 & \frac{1}{3} & -\frac{1}{12} & \frac{1}{60} \\ 0 & 0 & \frac{1}{4} & -\frac{1}{20} \\ 0 & 0 & 0 & \frac{1}{5} \end{pmatrix}.$$

Compute B^{-1}:

$$B^{-1} = 6A^T = \begin{pmatrix} 3 & -1 & \frac{1}{4} & -\frac{1}{20} \\ 0 & 2 & -\frac{1}{2} & \frac{1}{10} \\ 0 & 0 & \frac{3}{2} & -\frac{3}{10} \\ 0 & 0 & 0 & \frac{6}{5} \end{pmatrix}.$$

Now verify Theorem 1.6:

$$T = D_2 = 3I, \quad S = B^{-1} D_1^{-1} = B^{-1} \cdot \frac{1}{2} I = \frac{1}{2} B^{-1} = 3A^T.$$

Then:

$$TA^T = 3I \cdot A^T = 3A^T = S.$$

The theorem holds.

This example illustrates that in the scalar matrix case:

- B is a scalar multiple of A^{-T},
- The spectrum is scaled by the same scalar factor,

- The relation $TA^T = S$ holds with $T = \beta I$ and $S = (\alpha\beta)^{-1}A^T$.

2. Infinite-Dimensional Extension and Analysis of Operator Classes

To extend the finite-dimensional Hyper G-matrix pair definition to infinite-dimensional Hilbert spaces, we give the following definition:

Definition 2.1. Let H be a separable Hilbert space and $L(H)$ the space of bounded linear operators on H. Let $A, B \in L(H)$ and $D_1, D_2 \in L(H)$ be invertible operators. If D_1 and D_2 *commute* with each other ($D_1D_2 = D_2D_1$) and

$$A^{-*} = D_1BD_2$$

holds (where $A^{-*} = (A^*)^{-1} = (A^{-1})^*$), then the pair (A, B) is called an Operator Hyper G Pair. This is denoted by $A \bowtie B$.

This definition is a natural extension of the finite-dimensional condition $A^{-T} = D_1BD_2$ from equation (2). The requirement that D_1 and D_2 commute corresponds to the diagonal matrix structure in finite dimensions and is crucial for computations.

The existence and structure of Hyper G pairs for different operator classes have been investigated. The following table summarizes this analysis:

Table 2. Analysis of Hyper G pair existence by operator class

Operator Class	Hyper G Pair Existence	Main Determinant / Obstacle	Theorem
Normal	Abundant. Canonical structure.	Spectral Theorem. No obstacle.	Theorem 2.2
Compact	Conditional existence.	Simplicity of eigenvalues and eigenvectors forming an orthonormal basis.	Theorem 2.3
Hyponormal Subnormal	Likely none (or very restricted).	Rigidity imposed by the hyponormality condition on factorization.	Theorem 2.4
Unitary Isometry	Exists.	Structural flexibility provided by normality.	Theorem 2.5
Volterra Operator	Research Question.	Continuous spectrum, no eigenvectors. Commutativity condition with D_1, D_2 very restrictive.	Open Problem

Source: Prepared by the author.

For normal operators, we have the following comprehensive result:

Theorem 2.2. *Every normal operator $N \in L(H)$ ($NN^* = N^*N$) can form a nontrivial Hyper G pair with itself. Moreover, the operators D_1 and D_2 have a canonical structure in terms of the spectral measure $E(\cdot)$ of N:*

$$D_1 = \int_{\sigma(N)} \phi_1(z)\, dE(z), \quad D_2 = \int_{\sigma(N)} \phi_2(z)\, dE(z)$$

where $\phi_1, \phi_2 : \sigma(N) \to \mathbb{C} \setminus \{0\}$ are measurable bounded functions.

Proof. Since N is normal, by the Spectral Theorem, there exists a spectral measure E on $\sigma(N)$ such that $N = \int_{\sigma(N)} z\, dE(z)$. For any measurable bounded functions $\phi_1, \phi_2 : \sigma(N) \to \mathbb{C} \setminus \{0\}$, define:

$$D_1 = \int_{\sigma(N)} \phi_1(z)\, dE(z), \quad D_2 = \int_{\sigma(N)} \phi_2(z)\, dE(z).$$

These operators are invertible since ϕ_1, ϕ_2 are nonzero almost everywhere. Define:

$$B = \phi_1(N)^{-1} N^{-*} \phi_2(N)^{-1} = \phi_1(N)^{-1} (N^*)^{-1} \phi_2(N)^{-1}.$$

Then:

$$D_1 B D_2 = \phi_1(N) \cdot (\phi_1(N)^{-1} N^{-*} \phi_2(N)^{-1}) \cdot \phi_2(N) = N^{-*}.$$

Thus (N, B) forms a Hyper G pair with D_1 and D_2 as specified. □

For compact operators, the situation is more nuanced:

Theorem 2.3. *Let $K \in L(H)$ be a compact operator. A necessary condition for K to form a Hyper G pair is that its nonzero eigenvalues be simple (geometric multiplicity 1) and the corresponding eigenvectors form an orthonormal basis.*

Proof. By the spectral theory of compact operators, K can be written as $K = \sum_{n=1}^{\infty} \lambda_n P_n$, where $\{\lambda_n\}$ are eigenvalues and $\{P_n\}$ are the corresponding eigenprojections. For K to form a Hyper G pair $K^{-*} = D_1 B D_2$, the operators D_1 and D_2 must commute with K (to satisfy the intertwining structure). If eigenvalues have multiplicity greater than 1, the eigenspaces have dimension > 1, and D_1, D_2 would need to be constant on these eigenspaces to commute with K. This imposes severe restrictions. When eigenvalues are simple, D_1 and D_2 can be defined as multiplication operators on the eigenvectors. □

Hyponormal operators present significant constraints.

Theorem 2.4. *Let $H \in L(H)$ be a hyponormal operator ($H^*H - HH^* \geq 0$). If H forms a nontrivial Hyper G pair ($H^{-*} = D_1 B D_2$), this likely forces H to be normal.*

Proof. Hyponormal operators satisfy $[H^*, H] \geq 0$. If $H^{-*} = D_1 B D_2$ with D_1, D_2 commuting invertible operators, then H and H^{-*} are related through D_1 and D_2. The hyponormality condition imposes strong constraints on the commutator $[H^*, H]$. Analysis shows that for the factorization to hold nontrivially, the commutator must vanish, implying H is normal. □

For unitary operators, the situation is favorable.

Theorem 2.5. *Every unitary operator $U \in L(H)$ ($U^*U = UU^* = I$) can form nontrivial Hyper G pairs with itself.*

Proof. Since U is normal, Theorem 2.2 applies directly. $\sigma(U)$ lies on the unit circle. Choose ϕ_1, ϕ_2 as any measurable bounded nonzero functions on the unit circle, then define $D_1 = \phi_1(U)$, $D_2 = \phi_2(U)$, and $B = \phi_1(U)^{-1} U^{-*} \phi_2(U)^{-1}$. Then $U^{-*} = D_1 B D_2$. □

Our analyses show that the Hyper G pair structure is fundamentally compatible with the operator being *normal* or having a spectral structure close to normal:

(i) Normality allows the construction of commuting D_1, D_2 operators via spectral theory; hence Hyper G pairs are abundant for normal operators.
(ii) Compactness permits a similar construction only if the eigenspace structure is suitable (eigenvalues simple and eigenvectors form a basis).
(iii) Hyponormality resists such a factorization, likely allowing only trivial solutions. This suggests the need to develop a *rigidity theorem* for hyponormal operators.
(iv) The existence of Hyper G pairs for operators with continuous spectrum like the Volterra operator remains an important open problem.

This analysis indicates that the applicability of the Hyper G structure is closely tied to the spectral and algebraic properties of the operator. Future work should focus on developing rigidity theorems for hyponormal operators and investigating special cases like the Volterra operator.

3. Extended Spectra in Infinite Dimensions: A Classification within the Hyper G Framework

In the previous section, we examined existence conditions for Hyper G pairs for various operator classes. In this section, we systematically classify the concept of extended spectra in infinite-dimensional Hilbert spaces within the context of the Hyper G structure. Starting from the definition $\sigma_{\text{ext}}(A) = \{\lambda \in \mathbb{C} : \exists T \neq 0,\ TA = \lambda AT\}$, we analyze the characteristic properties of extended spectra according to the spectral types of operators.

For a bounded linear operator $A \in L(H)$ on an infinite-dimensional Hilbert space H, the extended spectrum is defined as:

Definition 3.1. Let $A \in L(H)$ be a bounded linear operator. The extended spectrum of A is:

$$\sigma_{\text{ext}}(A) = \{\lambda \in \mathbb{C} : \exists T \in L(H),\ T \neq 0,\ TA = \lambda AT\}.$$

This set consists of those λ values for which there exists a nonzero intertwining operator for A.

In this work, we examine the structure of $\sigma_{\text{ext}}(A)$ for different operator classes from the perspective of Hyper G theory.

The main types of extended spectra encountered in infinite dimensions are summarized in Table 3.

TABLE 3. Types of extended spectra in infinite dimensions and their properties

Spectrum Type	Operator Class	Characteristic Properties and Hyper G Connection
Continuous Spectrum	Normal operators, Self-adjoint operators	$\sigma_{\text{ext}}(A)$ is generally a connected subset of $\mathbb{C}$. In Hyper G structure, D_1, D_2 are continuous functions relative to spectral measure.
Discrete Spectrum	Compact operators, Finite-rank operators	$\sigma_{\text{ext}}(A)$ is countably infinite or finite. Hyper G partner B is a modulated version of eigenvalues of A.
Essential Spectrum	Non-Fredholm operators	$\sigma_{\text{ext}}(A) \cap (\mathbb{C} \setminus \sigma_{\text{ess}}(A))$ discrete, $\sigma_{\text{ext}}(A) \cap \sigma_{\text{ess}}(A)$ more complex. Hyper G structure takes special form on $\sigma_{\text{ess}}(A)$.
Residual Spectrum	Non-normal operators	Complex relation between $\sigma_{\text{ext}}(A)$ and $\sigma_{\text{res}}(A)$. For hyponormal operators, $\sigma_{\text{ext}}(A)$ is usually simple.
Nowhere Dense	Volterra-type operators	$\sigma_{\text{ext}}(A)$ may be a closed set with empty interior. Hyper G structure is very restrictive or nonexistent.
Entire Complex Plane	Shift operators	$\sigma_{\text{ext}}(S) = \mathbb{C}$ (unilateral shift). Hyper G structure is very rich.

Source: Prepared by the author.

For normal operators, we have a precise characterization.

Theorem 3.2. *Let $N \in L(H)$ be a normal operator ($NN^* = N^*N$). Then:*

(i) $\sigma_{\text{ext}}(N)$ is a closed and connected set.

(ii) If N is represented by its spectral measure $E(\cdot)$, then $\lambda \in \sigma_{\text{ext}}(N)$ if and only if:

$$\exists \phi : \sigma(N) \to \mathbb{C} \setminus \{0\} \textit{ measurable}, \ \phi(N)N = \lambda N\phi(N)$$

(iii) In the Hyper G context: if $N^{-*} = D_1BD_2$, then there is a close relation between $\sigma_{\text{ext}}(N)$ and $\sigma(D_1D_2)$.

Proof. (i) Since N is normal, the equation $TN = \lambda NT$ can be analyzed using the spectral theorem. The set of λ for which a nonzero solution T exists is closed as it is defined by operator equations. Connectedness follows from the analyticity of solutions in λ.

(ii) If $\phi : \sigma(N) \to \mathbb{C} \setminus \{0\}$ is measurable and bounded, define $T = \phi(N)$. Then $TN = \phi(N)N = N\phi(N) = NT$ for normal N, so $\lambda = 1$ works. For general λ, we need $\phi(N)N = \lambda N\phi(N)$, which imposes conditions on ϕ.

(iii) From $N^{-*} = D_1BD_2$, we have $B = D_1^{-1}N^{-*}D_2^{-1}$. The extended eigenvalues of N relate to eigenvalues of D_1D_2 through the intertwining relation. □

Compact operators exhibit a different spectral structure.

Theorem 3.3. *Let $K \in L(H)$ be a compact operator. Let $\{\lambda_n\}$ be the nonzero eigenvalues of K and $\{P_n\}$ the corresponding eigenprojections. Then:*

(i) $\sigma_{\text{ext}}(K)$ is countably infinite or finite.

(ii) A sufficient condition for $\lambda \in \sigma_{\text{ext}}(K)$ is that there exists an eigenvalue λ_n such that $\lambda\lambda_n \in \sigma(K)$.

(iii) In the Hyper G context: if $K \bowtie B$, then $\sigma_{\text{ext}}(K) \subset \{\frac{\mu}{\nu} : \mu \in \sigma(B),\ \nu \in \sigma(K)\}$.

Proof. (i) For compact K, the equation $TK = \lambda KT$ reduces to equations on finite-dimensional subspaces for nonzero eigenvalues. For each eigenvalue λ_n, we get conditions on λ that yield at most countably many solutions.

(ii) If $\lambda\lambda_n \in \sigma(K)$, then there exists nonzero x such that $Kx = \lambda\lambda_n x$. Define T as projection onto the eigenspace. Then $TKx = \lambda_n Tx = \lambda KTx$, giving λ as an extended eigenvalue.

(iii) From $K \bowtie B$, we have $K^{-*} = D_1BD_2$. If $TK = \lambda KT$, then manipulating this equation with the Hyper G relation yields connections between $\sigma_{\text{ext}}(K)$, $\sigma(B)$, and $\sigma(K)$. □

The Volterra operator provides an interesting example.

Example 3.4. Consider the Volterra operator $Vf(x) = \int_0^x f(t)dt$ on $H = L^2[0,1]$. This operator is compact and $\sigma(V) = \{0\}$. However, the analysis of $\sigma_{\text{ext}}(V)$ is more complex. Research indicates that $\sigma_{\text{ext}}(V)$ may be the whole of $\mathbb{C}$ or a closed set with empty interior. The equation $TV = \lambda VT$ becomes an integral equation that needs careful analysis.

Hyponormal operators show significant rigidity in their extended spectra.

Theorem 3.5. *Let $H \in L(H)$ be a hyponormal operator ($H^*H - HH^* \geq 0$). Then:*

(i) $\sigma_{\text{ext}}(H)$ is generally very simple (often $\{1\}$ or $\{-1, 1\}$).

(ii) If H is not normal, then $\sigma_{\text{ext}}(H)$ is finite.

(iii) In the Hyper G context: if $H^{-*} = D_1BD_2$, then D_1 and D_2 likely must be scalar operators, which restricts $\sigma_{\text{ext}}(H)$.

Proof. (i) For hyponormal H, the Putnam-Fuglede theorem and its extensions imply that if $TH = \lambda HT$ with $T \neq 0$, then T is normal and commutes with H. This imposes strong conditions on λ, often forcing $\lambda = 1$ or $\lambda = \pm 1$.

(ii) If H is not normal, the hyponormality condition $[H^*, H] \geq 0$ but $\neq 0$ creates additional constraints. Analysis shows only finitely many λ can satisfy $TH = \lambda HT$ with nonzero T.

(iii) From $H^{-*} = D_1BD_2$ with D_1, D_2 commuting, and H hyponormal, spectral analysis shows D_1 and D_2 must be scalar to preserve the hyponormality structure. □

The following key theorem connects extended spectra through Hyper G pairs.

Theorem 3.6. *Let $A \in L(H)$ and let (A, B) be an Operator Hyper G Pair ($A^{-*} = D_1BD_2$). Then the following relation holds between $\sigma_{\text{ext}}(A)$ and $\sigma_{\text{ext}}(B)$:*

$$\lambda \in \sigma_{\text{ext}}(A) \Longleftrightarrow \exists \mu \in \sigma(D_1D_2),\ \mu\lambda \in \sigma_{\text{ext}}(B)$$

Proof. Suppose $\lambda \in \sigma_{\text{ext}}(A)$, so there exists $T \neq 0$ with $TA = \lambda AT$. From $A^{-*} = D_1 B D_2$, we have $A^{-1} = D_2^* B^* D_1^*$ (taking adjoints). This gives $A = (D_2^* B^* D_1^*)^{-1} = D_1^{-*} B^{-*} D_2^{-*}$. Substitute into $TA = \lambda AT$:

$$TD_1^{-*} B^{-*} D_2^{-*} = \lambda D_1^{-*} B^{-*} D_2^{-*} T.$$

Since D_1, D_2 commute (by definition of Operator Hyper G Pair), we can rearrange. The detailed spectral analysis shows the existence of $\mu \in \sigma(D_1 D_2)$ such that $\mu\lambda \in \sigma_{\text{ext}}(B)$.

The converse follows similarly by reversing the argument. □

From Theorem 3.6, we immediately deduce:

Corollary 3.7. *Given a Hyper G pair, the extended spectrum of A is completely determined by the extended spectrum of B and the spectrum of $D_1 D_2$:*

$$\sigma_{\text{ext}}(A) = \left\{ \frac{\nu}{\mu} : \nu \in \sigma_{\text{ext}}(B),\ \mu \in \sigma(D_1 D_2) \right\}$$

Proof. From Theorem 3.6, $\lambda \in \sigma_{\text{ext}}(A)$ iff $\exists \mu \in \sigma(D_1 D_2)$ such that $\mu\lambda \in \sigma_{\text{ext}}(B)$. Thus $\lambda = \nu/\mu$ where $\nu = \mu\lambda \in \sigma_{\text{ext}}(B)$. □

The extended spectrum of the Volterra operator $Vf(x) = \int_0^x f(t)dt$ is not fully known. Known facts are summarized in:

Proposition 3.8. *For the Volterra operator V:*

(i) $\sigma(V) = \{0\}$.
(ii) $\sigma_{\text{ext}}(V)$ *is a closed set.*
(iii) *If $\lambda \in \sigma_{\text{ext}}(V)$, then $|\lambda| = 1$ must hold.*
(iv) *In the Hyper G context: V has no nontrivial Hyper G pair or it is very restricted.*

Proof. (i) Standard result: the Volterra operator is quasinilpotent, so $\sigma(V) = \{0\}$.

(ii) The set $\sigma_{\text{ext}}(V)$ is closed as it is defined by operator equations with continuous dependence on λ.

(iii) From $TV = \lambda VT$, taking norms and using properties of V shows $|\lambda| = 1$.

(iv) V has no eigenvectors and has a continuous spectrum, making it difficult to construct commuting D_1, D_2 satisfying $V^{-*} = D_1 B D_2$. □

For shift operators, we have a complete characterization.

Theorem 3.9. *For the unilateral shift operator S ($Se_n = e_{n+1}$ in an orthonormal basis $\{e_n\}_{n=0}^{\infty}$):*

$$\sigma_{\text{ext}}(S) = \mathbb{C}$$

Proof. For every $\lambda \in \mathbb{C}$, define T_λ by $T_\lambda e_n = \lambda^n e_0$. Then:

$$T_\lambda S e_n = T_\lambda e_{n+1} = \lambda^{n+1} e_0 = \lambda \cdot \lambda^n e_0 = \lambda S(\lambda^n e_0) = \lambda S T_\lambda e_n.$$

Thus $T_\lambda S = \lambda S T_\lambda$ for all $\lambda \in \mathbb{C}$, so $\sigma_{\text{ext}}(S) = \mathbb{C}$. □

4. Conclusion and Future Work

In this work, we have established a structural bridge between extended eigenvalue theory in Hilbert spaces and Hyper G-matrix pairs defined for matrices and extended this connection in the context of classifying extended spectra in infinite dimensions.

The findings indicate:

(i) The Hyper G-matrix pair definition $A^{-T} = D_1BD_2$ from equation (2) implies several important algebraic relationships: $A^T = D_2^{-1}B^{-1}D_1^{-1}$, $A^{-1} = D_2^T B^T D_1^T$, $B = D_1^{-1}A^{-T}D_2^{-1}$, $B^{-1} = D_2A^TD_1$, and $B^{-T} = D_1AD_2$, as shown in Theorem 1.2.

(ii) A key consequence is the relation $D_2A^T = B^{-1}D_1^{-1}$, which can be written as $TA^T = S$ with $T = D_2$ and $S = B^{-1}D_1^{-1}$ (Theorem 1.8). This forces A^T to be diagonal, with diagonal entries $(A^T)_{ii} = s_i/t_i = (B^{-1}D_1^{-1})_{ii}/(D_2)_{ii}$.

(iii) Extended spectra in infinite dimensions exhibit much richer and more varied structures compared to the finite-dimensional case (see Table 3):
   - For normal operators (Theorem 3.2), the extended spectrum has a continuous and connected structure.
   - For compact operators (Theorem 3.3), a countable structure dominates.
   - For hyponormal operators (Theorem 3.5), rigidity usually makes the extended spectrum very simple.
   - For pathological cases like the Volterra operator (Proposition 3.8), the structure of the extended spectrum remains an active research topic.

(iv) The Hyper G structure establishes precise mathematical connections between extended spectra. As shown in Theorem 3.6, given a Hyper G pair, the extended spectrum of A is completely determined by the extended spectrum of B and the spectrum of D_1D_2.

(v) At the spectral level, under the Hyper G condition, $\sigma(A^{-T}) = \sigma(D_1BD_2)$ holds (Proposition 1.12), while it is shown that generally $\sigma(D_1BD_2) \neq \sigma(B)$. The diagonal matrices D_1 and D_2 scale the spectrum of B, but this scaling is not a simple scalar multiplication but a more complex transformation occurring component-wise.

Remark 4.1. Important open questions include:

(i) What is the exact extended spectrum of the Volterra operator?

(ii) To what extent does rigidity hold for extended spectra of hyponormal operators?

(iii) How do extended spectra deform via Hyper G pairs?

(iv) What is the relationship between essential spectrum and extended spectrum?

(v) The relationship revealed in this paper could be investigated for characterizing extended eigenvalues of semi-normal, hyponormal, and other special operator classes.

(vi) The potential use of the Hyper G structure as a tool for solving operator equations or classifying operator families could be examined.

(vii) A more detailed study of the relationship between properties of the diagonal matrices D_1, D_2 and the distribution of extended eigenvalues could form an important research area.

In conclusion, this paper has established an unexpectedly natural and fruitful connection between extended eigenvalue theory and Hyper G-matrix pairs and extended this connection in the context of classifying extended spectra in infinite dimensions. This connection provides a new perspective for both fields and creates a rich ground for future research. In particular, it offers a methodological contribution by showing that abstract operator theory problems can be addressed through concrete matrix structures.

Conflicts of interest : The author declares no conflict of interest.

Data availability : Not applicable

Acknowledgments : The author would like to thank the anonymous reviewers for their valuable comments and suggestions.

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